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Article

Capturing changes in factor effectivity with estimates of a CES production function with flexible trends

Capturing Changes in Factor Effectivity with Estimates of a CES Production Function with Flexible Trends[1]

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Abstract

Constant Elasticity of Substitution production function allows us, compared with a Cobb-Douglas function, to model different efficiency trends of labor and capital. In this article, we explore the efficiency trends of labor and capital in supply systems for the private and public sectors in Slovakia independently. If a single exponential trend for technical progress is used, the Cobb-Douglas production function can be used for the private sector and Leontieff production function for the public sector. We, however find a CES function with separate trends based on Box-Cox transformations outperforms capturing technological progress by models with a single exponential trend. Labor and capital efficiency gains in our preferred model are converging downwards to a positive constant for labor and increasing over time for capital in the private sector, whereas they are gradually decreasing for both factors in the public sector. The elasticity of substitution between labor and capital is significantly greater than zero, but also lower than 0.5 in preferred models for both sectors.

Keywords: *production function, time trend, Box-Cox transformation*

JEL Classification: O47, E10

Introduction

Production functions are a standard way of modelling the production process on aggregate level. They determine output or value added from production factors – labor and capital, and, incorporate technical progress. Demand functions

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[1] This article presents the opinions of its author only, not the official viewpoints of National Bank of Slovakia.

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for production functions can be derived from the production function, determining needed volume of each factor from output and prices. These functions can be combined in such way, that the output drops out and the ratio of volumes of production factors is determined by the ratio of their prices – so called relative demand function (for its derivation for CES function, see Oaxaca). The easier is the substitution of one factor with the other one, the more the ratio of volumes of production factors changes with the ratio of their prices. The respective parameter is elasticity of substitution[2] is inherent to every production function. The limiting case is complementary or Leontieff[3] production function, where a fixed ratio of production factors is needed to produce a single unit of output. The elasticity of substitution in this case is zero – ratio of factor volumes is independent of prices, because it is fixed by definition. Another useful case, often encountered in practice, is the case of unit elasticity of substitution. In this case, output is a power function (so called Cobb-Douglas production function) of production factors. Moreover, if we associate costs with factors, so that labor costs are a product of wage (price of labor) and labor volume and capital costs are a product of capita volume and user costs of capital, unit elasticity of substitution leads to a constant share of labor costs on output (this is true also for capital costs). Parameters of the production function can also be inferred from these ratios. Arrow et al. (1961) derived a general form of CES (constant elasticity of substitution) production function, allowing any positive value for that parameter apart from unity. Estimations in this commentary are performed using this function.

Another important parameter of a production function is return to scale, describing, by how much output increases, if the inputs are multiplied by certain quotient. For traditional technologies, constant returns to scale are plausible, so that the output is multiplied by the same amount as inputs are. For advanced technologies, where there is significant learning by doing, or network effects, output can be multiplied by higher quotient than the inputs (increasing returns to scale). In this article, we assume constant returns to scale in order to avoid ambiguities in estimation, as the aforementioned advanced technologies constitute only a tiny fraction of the economy in Slovakia.

Production functions are often incorporated into big econometric models, Livermore (2004), Reľovský and Široká (2009) use the Cobb-Douglas production function for Slovakia, Benk et al. (2006) use the CES production function for Hungary. Estimation of supply systems, like Willman (2002) or Klump, McAdam

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[2] The text in this article sums up the properties of production functions in non-technical terms. Further information and technical details about production function and elasticity of substitution can be found e.g. in Chiang (1984).

[3] Leontieff function refers to the complementary production function, as in Allen (1968). This production function is the key assumption of input-output model.

and Willman (2004) narrows the focus to parameters and properties of supply side only, with production functions being central part of those supply systems. The concept of production function can be applied in microeconomics as well, where special methods are sometimes necessary (for example Levinsohn and Petrin, 2003). There are several refinements and generalizations of production functions at the macro level: nested production functions use more than two factors and they arranged into a CES function in a CES function as in León-Ledesma, McAdam and Willman (2011) or Sato (1967). This approach allows the elasticity of substitution vary between production factors and their aggregates. Another possibility is a non-homothetic production function where output level and technical progress have impact on the factor input ratio (Sato, 1975).

Beside of applying traditional production functions at macro level, there exists an analytical approach by Kotěšovcová, Mihola and Wawrosz (2017), who define a typology of output growth based on indicators of total factor productivity TFP and total intensity of factors TIF. By using geometric mean of factors, this approach, however, does impose the unit elasticity of substitution, like the Cobb--Douglas production function.

This work was motivated by the opinion, that the Cobb-Douglas production function imposes too many important assumptions upon the data and that the "nice" results from that function reflect the untested assumptions rather than the reality. The assumption about constant factor shares in output is most easily disproved, but there are others as well. Our informal opinions are supported by Felipe and Holz (2005), who study with Monte Carlo simulations, why the Cobb-Douglas production function on aggregate level gives "nice" results that have little information value. The aim of our work has been studying the supply side of Slovak economy using methods capable of capturing its unique features, with results determined by data instead of restrictive assumptions. We thus decided to use a batch of more general approaches. Beside of the presented system estimation, we tried out the nested production function and integrating the factor utilization into production function, in line with Roeger (2006). Those last two approaches, however, did not lead to meaningful results. Our estimates show that the results depend upon the exact specification of production function and assumption whether the rate of technical progress is constant in time or not and that the preferred model shall be chosen according to a statistical criterion.

Our work adds to the existing literature applying production functions to Slovak data in two aspects: (i) it applies a production function with general factor specific trends that can be concave, linear or convex and (ii) derives and applies the relative demand function for the estimation with aforementioned flexible trends. Apart of these innovations, we estimate the supply system separately for private and public sectors, as the relations between production factors and output are different in those sectors.

The rest of the article is structured as follows: after discussing the properties of most widely used production functions (Section 1) and the choice of the supply system (Section 2), we present the technical aspects of estimation (Section 3) and results (Section 4), we extract the efficiency gains (Section 5) and conclude.

1. Properties of Cobb-Douglas and CES Production Functions and Their Practical Aspects

Cobb-Douglas production function is easy to estimate in logarithms by restricted or unrestricted ordinary least squares. It is invariant to units used, these impact the constant term only. Less known problem is that the constant term in log-linear models needs correction, because the mean of exponentiated residuals with zero mean is not equal to unity and depends upon the residual sum of squares (see Newman, 1993). Modelling technical progress is in practical application mostly confined to exponential trend, although more analytical ways are possible (see Willman, 2002).

CES production function – its functional form of the CES production function is non-linear. It thus requires an estimation technique based on numerical optimization and starting values. Some parameters may converge outside of the definition domain of the function during the optimization process and require calibration. It is not invariant to changes in units, this is best addressed by using base values and indices. The main point of this approach is to introduce base period in which output, capital and labor assume base values. The base values were determined as the geometric means of respective variables and the base period was set to period, when these values occurred in the sample. By dividing the economic variables with their base values (using indices) and shifting the time trend we transformed the production function so that both the left-hand side and the variables on the right-hand side are simultaneously equal to unity[4] in the base time period. In this period the time trend is equal to zero and efficiency trends are equal to one. The CES function allows for separate efficiency trends associated with each production factor. There are more alternatives how to do this, but the state of the art is to use Box-Cox transformations (see below), that are fairly general. Estimates with such separate efficiency trends can mimic the technical progress better and reveal interesting information.

[4] If the variables on the left-hand side and right-hand side assume their geometric means in the same period and this period is chosen as the base period, the parameter A_0 will be identically equal to unity.

Estimated parameters in both functions may be at odds with the ratio or employee compensations to output. Thus, it may be preferable to use calibrated value of the parameter in question and estimate the rest. Both functions are prone to yield parameters consistent with increasing returns to scale when estimated unconstrained. However, this is plausible for neither private nor public sector in Slovakia as a whole and is most likely caused by parameter variation in the business cycle (see Cette et al., 2015). The proposed solution, inclusion of capacity utilization in production function, does not work in our case, because this indicator does not track the business cycle well enough for Slovakia (this was tried out in previous version of our research). We addressed this problem by a restriction leading to constant returns to scale and using a dummy variable in our estimations.[5]

2 Choice of Supply System

We use the CES production function, as the factor shares in output are not constant. The analysis in this commentary is based on four estimates of supply systems. System estimations have the advantage that the elasticity of substitution appears in more than one equation and these cross-equation constraints capture more information about this parameter than single equation estimates. There are at least two types of systems that were used in the European Central Bank: a system with production function, relative demand function and price function (Willman, 2002) and a system with production function and factor demand functions (Klump, McAdam and Willman, 2004). We wanted to estimate the newer system for Slovakia originally, but the factor demand functions proved problematic (the residual from the production function was transferred into the factor demand function with considerable amplification for low values of elasticity of substitution). The relative factor demand function proved more suitable, since it does not contain output and does not suffer from the aforementioned problem. Note that Willman (2002) uses a relative demand function with the ratio of costs rather than volumes, but we found it more appropriate to use volumes on the left-hand side and prices on the right-hand side rather than using both on the left-hand side as we wanted to eliminate potential bias in estimation. The system in Willman contained a price equation with price being the function of marginal product of labor, wage rate and markup. In case of Slovakia, however, price setting in the sample period

[5] We could, of course, use a more complex indicator of output gap, as in Ódor and Jurášeková Kucserová (2014), but such composite indicator would mimic the variation of left-hand side variable, and its inclusion would practically introduce tautology into the model. It could also incorrectly pick the variation of output that was due to varying inputs. On the other side, leaving the equation without correction and accepting the increasing returns to scale would be unrealistic in our case. Because of these cosiderations, we chose a simple dummy.

was heavily influenced by the transition from centrally planned economy to market economy and apart from that, the markup computed from Slovak data was very erratic. We thus decided to drop the price equation and estimate systems with the production function and relative demand function in each only.

3. Technical Details of Estimation[6]

The supply system consists of a production function and a relative demand function. We denote *Q* the value added, *K* the net capital stock and *L* the employment in hours and *t* the time trend, the other symbols being the parameters, while the letters with bars denote base values. The CES production function is used, since the factor shares in output are not constant.[7] Technical progress for both sectors is captured twofold: i) in a system with a simple exponential trend and ii) with a Box-Cox transformation.[8] The equations (1) and (2), resp. (1) and (3) are combined into a single equation in estimations.

$$\left(\frac{Q}{\bar{Q}}\right)=A_t\left[a\check{K}^{\rho}+\left(1-a\right)\check{L}^{\rho}\right]^{\frac{1}{\rho}}, \text{ where } \rho=\frac{\sigma-1}{\sigma} \tag{1}$$

For the two versions with exponential trends,

$$A_t=A_0\exp\left[\psi\left(t-\bar{t}\right)\right],\ \check{K}=\left(\frac{K}{\bar{K}}\right) \text{ and } \check{L}=\left(\frac{L}{\bar{L}}\right) \tag{2}$$

The parameter A_t can be interpreted as a measure of the total factor productivity in this case.

For the two versions with Box-Cox transformation,

$$A_t=A_0,\ \check{K}=\left(\frac{K}{\bar{K}}\right)exp\left\{\frac{\bar{t}\,\gamma_1}{\lambda_1}\left[\left(\frac{t}{\bar{t}}\right)^{\lambda_1}-1\right]\right\} \text{ and } \check{L}=\left(\frac{L}{\bar{L}}\right)exp\left\{\frac{\bar{t}\,\gamma_2}{\lambda_2}\left[\left(\frac{t}{\bar{t}}\right)^{\lambda_2}-1\right]\right\} \tag{3}$$

The second equation in each system is a dynamized relative demand function. This function relates the ratio of factor volumes to the ratio of factor prices and separate time trends, if they are used in the production function.[9] The relative demand function for the case with separate trends for production factors is derived in the Appendix.

[6] We use the symbol log(x) for natural logarithms in equations, in line with the majority of the literature and the E-views package. The notation ln(x) is more widely used in physics and engeneering.

[7] Variability of labor cost ratio to value added is observable in data.

[8] Technical details of Box-Cox transformation are available e.g. in Klump, McAdam and Willman (2004).

$$log\left(\frac{K}{L}\right)=C_0+\nu log\left(\frac{K_{t-1}}{L_{t-1}}\right)+(1-\nu)\sigma log\left(\frac{w}{q}\right)+$$
$$+(1-\nu)(\sigma-1)\left\{\frac{\bar{t}\,\gamma_1}{\lambda_1}\left[\left(\frac{t}{\bar{t}}\right)^{\lambda_1}-1\right]-\frac{\bar{t}\,\gamma_2}{\lambda_2}\left[\left(\frac{t}{\bar{t}}\right)^{\lambda_2}-1\right]\right\} \tag{4}$$

The CES production function has a constant term A_0, that is by construction bound to be near unity. The parameter a distributes weights between capital and labor (labor weight is denoted as $1 - a$). The parameter σ is the elasticity of substitution, which measures the ease of substitution between capital and labor, and vice versa, and the elasticity of ratio of factor volumes to ratio of factor prices. The parameter ψ is the annual rate of technical progress when it is modeled as an exponential trend. Parameters γ_i and λ_i determine the shape of separate trends for capital (i = 1) and labor (i = 2). The parameter C_0 is the constant term in the relative demand function and v is the parameter of autoregressive term in this equation and measures the inertia of the ratio of factor volumes. The base period relates to the date to which indices used in the production function are fixed

The system was partially calibrated[10] and partially estimated by Full information maximum likelihood. The starting values were chosen either to a value consistent with the use if indices in CES function (A_0), consistent with earlier research (σ, CES function for Slovakia was estimated by Benčík (2008) earlier) and small but non-zero values for parameters describing technical progress. There was a dummy variable in the production function for private sector, accounting for overheating economy before 2009 (in order to overcome the aforementioned problem with the returns to scale) and another one in the relative demand function for public sector, rectifying data issues. [11] The estimation sample begun in 1995 and ended in 2018Q4.

[9] In systems with single exponential trend, the last term is not used, the complete equation is used in systems with Box-Cox transformations.

[10] Parameter a (distribution parameter) was calibrated according to the share of compensation of employees in the value added in the respective sector; elasticity of substitution was calibrated to a small but non-zero value for the public sector estimate with exponential trend. The value used is statistically not significantly different from zero, but mathematically is, so that it allows the rest of the system to be estimated. In standard estimation, parameter σ converged out of domain of definition of the CES production function in (1) and the estimation crashed.

[11] Both dummies were introduced in a way that they modify the constant A resp. A_0 or C_0 so that $A_0 \rightarrow A_0$ (1 + dummy). This is omitted in the equations in order to keep the explanation simpler. The dummy for public sector assumed value 1 in 1997Q3 and –0.5 in 1997Q4, zero otherwise and entered the relative demand function. The reason for inclusion of this dummy was a inexplicably high value of labor in 1997Q3. We could discard the whole year 1997 as an alternative, but the sample is short already and we wanted to capture as much of the expansion in 1997 and 1998 as possible. The dummy for private sector assumed value 1 for the period 2007Q1 to 2008Q4, zero otherwise and entered the production function.

The relative demand function uses so-called Koyck transformation, assuming that the ratio of factor volumes reacts to changes in ratio of factor prices with a lag in a process of continuous correction. The resulting gaps, contrary to Morvay and Hudcovský (2018), are only of short-term nature. The introduction of Koyck transformation moved the residuals of the relative demand equations nearer to white noise and also improved fit compared to static versions. The values of Durbin Watson test indicated severe positive autocorrelation for production function (not reported), but as this is the result of parameter instability in business cycle rather than dynamic misspecification, we just included the dummy for overheating and did not try any further remedies. The estimated values of distribution parameter (a) were inconsistent with stylized facts, so they were calibrated. It is evident from the Table 1 that the estimates of elasticity of substitution are tightly connected with the specification of technical progress.

4. Results[12]

The main results are in the following Table:

Table 1

Results of Estimation for Various Versions of the Supply System

Parameter/Version	Private sector – exponential trend	Private sector – Box-Cox transformations	Public sector – exponential trend	Public sector – Box-Cox transformations
A_0	0.971***	0.987***	0.973***	1***
a	0.6 c	0.6 c	0.3 c	0.3 c
σ	1.081***	0.304***	0.01 c	0.127**
ψ	0.026***		0.021***	
γ_1		0.021***		0.018***
λ_1		0.950***		–0.232
γ_2		0.034***		0.026***
λ_2		0.488***		0.071
c_0	0.044*	–0.129**	–0.115***	–0.391***
v	0.933***	0.784***	0.881***	0.539***
Dummy – private s.	0.122***	0.109***		
Dummy – public s.			–0.258***	–0.21***
Base period	2006Q1	2006Q1	2004Q2	2004Q2
Rsq – prod. function	0.988	0.991	0.829	0.901
Rsq – rel. demand	0.984	0.987	0.899	0.926
Log likelihood	–1 586.6	–1 570.3	–1 423.1	–1 382
Schwarz criterion	33.69	33.49	30.20	29.53

Note: Parameters denoted */**/***/c are significant at 10%/5%/1%/calibrated.

Source: Own computations.

[12] Complete estimation output is beyond the scale of this article and is available upon request.

Whereas for the private sector, the elasticity of substitution is high, practically equal to unity for the version with exponential trend, it is much lower for the version with Box-Cox transformation. If we assumed that the technical progress is sufficiently captured with an exponential trend, we could use the Cobb-Douglas production function instead of the CES function. This finding is consistent with Willman (2002). We have in fact estimated a restricted version of Cobb-Douglas function (not reported) and constructed another one using parameters from the CES estimate in the first column of Table 1. We computed the residuals form these functions and transformed them so that they were comparable with the residuals from the CES function. We then carried out Kolmogorov-Smirnov test, comparing the distribution of the residuals from the CES function with either version of the Cobb-Douglas function. In both cases, the test has shown that the distributions are identical. Thus, we positively proved that the CES function with exponential trend can be replaced with the Cobb-Douglas function. However, if we introduce efficiency trends for both production factors, the elasticity of substitution decreases so that the Cobb-Douglas function can be no longer used. Values of elasticity of substitution between zero and unity for systems with Box-Cox transformations are presented in Klump, McAdam and Willman (2004). For the model with separate trends, the resulting production function is nearer to the Leontieff function than the Cobb-Douglas function in this case but is distinct from both limiting cases. The elasticity of substitution changed because the more general form of Box-Cox transformations allows greater flexibility of time trends compared with single exponential trends. The explained variance is then redistributed among variables because of this increased flexibility, leading to changes in estimates of elasticity of substitution. The Schwarz criterion is marginally higher for the model with Box-Cox transformations and we find the version with Box-Cox transformation superior as well because it reveals new information about efficiency trend of every production factor.

For the public sector, the value of the elasticity of substitution form the version with exponential trend converged near zero, showing that we could use Leontieff function instead, Note the low value of R squared for the relative demand – the term with the ratio of prices almost fully vanished in this estimation. However, if we introduce separate efficiency trends, the fit in both equations improves significantly. The superiority of the estimate with Box-Cox transformation is confirmed by the Schwarz criterion. The resulting elasticity of substitution in this case is lower than that for private sector, but still different from zero at 5% significance level. Note that the summary statistics are better for the system with Box-Cox transformation, even if it contains insignificant parameters. This means that the assumption of technical progress as an exponential progress is unsuitable for the public sector. In next section, we will explore the evolution of technical progress in detail.

5. Efficiency Trends as Implied By the Estimates of CES Production Function with Box-Cox Transformations

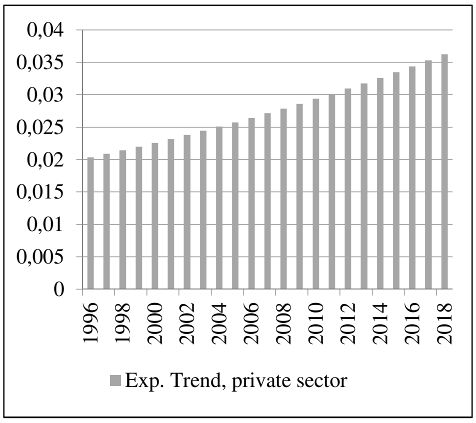
Technical progress causes the gradual rise of efficiency of production factors in time. It is modelled by functions of a linear time trend. These functions are continous, with continous first and second derivative. We explored versions with single exponential trend that averages various effects and describes technical progress with a single parameter and more complex versions using separate trend functions for every production factor. These functions have different properties. The exponential function is more restrictive, because it implies certain shape of resulting trend (that the absolute changes in efficiency will be higher at the end of the sample than at the beginning). Box-Cox transformations use the linear time trend and base period as arguments and allow the resulting trend function to be concave, linear or convex. They thus constitute flexible trends. They are more general than the exponential function and consequently, use two parameters for every factor. In cross-section studies (for example Caselli, 2017), production functions contain country-specific efficiency factors that are analogous to efficiency trends in this article. Efficiency, measured by a trend, increases in time in standard macroeconomic studies, but there are cases, when the efficiency trend is decreasing. For example, in studies with vintage production functions they are decreasing and capture the decreasing efficiency of capital vintages as they age. An example is the beta-decay function (Tokui, Inui and Kim, 2008). The function of beta-decay assumes value one when the capital is new and zero when it reaches its end of life. Sometimes the efficiency is measured without using a trend. The total factor productivity as a measure of efficiency can be computed from data for smaller units using mathematical programming (Balka, Barbero and Zofíode, 2020).

We present the efficiency trends from the production function. The possibility of such trends follows from the form of the CES function. The efficiency can be factor-specific in this setup, that is much more general than using single trend. The necessity to use them arises from the factor demand functions. A CES function without them implies factor demand functions that are restricted relative to the CES function with separate trends (see Footnote 9). It can be tested whether the new parameters are needed or not. In our case, the models with separate trends are preferred according to Schwartz criterion (compare the results in Table 1 for the exponential trend and Box-Cox transformations). It is important to test the whole systems rather than the production function only as the separate trends impact the factor demand more than the production function itself. By including the separate trends into the production function, we eliminated the restrictive nature of production function with a single trend. We thus allowed the model to follow stylized facts more closely, instead of imposing the results by restrictions, criticized by Felipe and Holz (2005). After the estimation, these trends were computed as $t_e = \exp\left[\psi\left(t - \bar{t}\right)\right]$ in the case with exponential trends and as $t_i = exp\left\{\frac{\bar{t}\,\gamma_i}{\lambda_i}\left[\left(\frac{t}{\bar{t}}\right)^{\lambda_i} - 1\right]\right\}$ with i = 1 for capital and i = 2 for labor in the case with Box-Cox transformations. All resulting trends were strictly increasing in time. In order to highlight the differences among them as well as properties of the functions used, we computed the annual absolute changes of these trends and present them in following four graphs. For easier judgement about the absolute changes, we remind the reader that the efficiency trends are equal to one in the base period (in 2006Q1 for the private sector and 2004Q2 for the public sector).

General efficiency trends for the simpler versions of supply system are shown in Figure 1a and 1b.

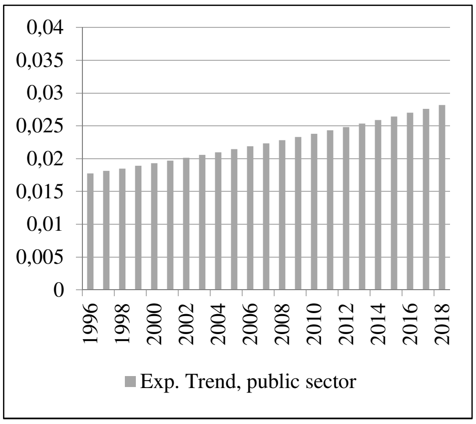
F i g u r e 1a

Annual Changes of General Efficiency Trend, Private Sector



F i g u r e 1b

Annual Changes of General Efficiency Trend, Public Sector



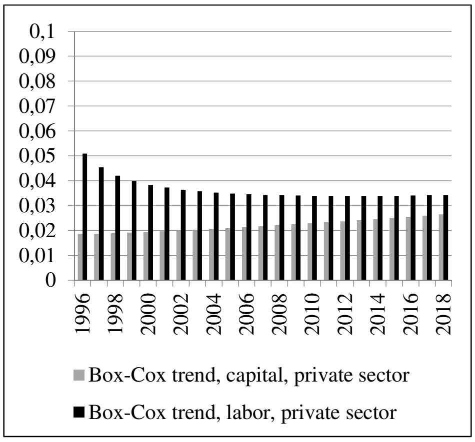
Source: Own computations.

Unsurprisingly, the technical progress is somewhat slower in the public sector. In both cases, the annual changes exhibit rising linear trend. This is determined by the properties of the exponential function (derivative of exponential function is a linear function). The exponential function has a clear interpretation that its result rises with a constant growth rate, but it can be too restrictive. As the Schwarz criterion of estimates for both sectors was lower for the version with exponential trend, we can assume that that was the case.

The efficiency trends for supply systems with Box-Cox transformations are shown in Figures 2a and 2b.

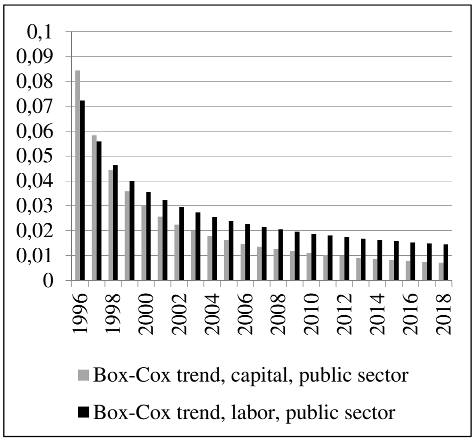
F i g u r e 2a

Annual Changes of Efficiency Trends, Private Sector



F i g u r e 2b

Annual Changes of Efficiency Trends, Public Sector



Source: Own computations.

For both sectors, efficiency of labor grows faster than the efficiency of capital in most periods. The curves encompassing the annual increases have distinctly different character from the previous case.

The efficiency of capital in private sector is most comparable to the general efficiency trend from Figure 1a, both because of increasing trend (implying a rising and convex efficiency level), even if the values are lower. The efficiency of labor starts with high values of 0.5, but converges quickly to 0.34, so that the changes are roughly constant last fifteen years, and the resulting trend level can be approximated by a straight line (in fact it bottoms in 2013 and rises marginally since then).

Both the annual efficiency changes for the public sector are strictly decreasing, implying concave efficiency level that somewhat resemble logarithmic function. The efficiency level rises most at the beginning of the sample.[13] The annual efficiency gains are higher than those in the private sector in the beginning of the sample, but far below them in the end of the sample,[14] probably converging to a value near zero for both labor and capital in future.

[13] We have considered a possibility that the high efficiency gains at the beginning of the sample are a result of a data issue or a different relation in general. We estimated the same system with sample starting in 1997 (not reported) and shifted base year. The parameters changed somewhat, but the shape of efficiency trend remained the same, the presented result is thus robust.

[14] If the efficiency gains for labor in public sector were comparable to those in private sector and efficiency gains for capital were subdued, one could argue that that is the result of roads, bridges and other infrastructure being added to public capital stock, as they are mostly used free of charge. However, we find this explanation unlikely since the situation is obviously different.

From the comparison of resulting trends for both versions and sectors it is evident that the technical progress is too complex phenomenon to be captured with a simple exponential trend.

In general, we expected that the efficiency gains for private sector will be larger than for the public sector, since the former is subject to Schumpeterian creative destruction and forced to evolve constantly. However, the results for the public sector pose either a stark warning that the increase in output since early 2000s did not match the increase in factor used, so that a significant gap in efficiency emerged or that there is a problem with measurement of factors, especially capital. We use net capital and this measure assumes that after the capital goods are depreciated, they vanish from the production process. Net capital actually decreased in the beginning of the sample. However, there is a high share of structures in the capital stock for public sector and these remain in use as long as they are physically intact, irrespective of depreciation. This phenomenon might lead to an incorrect path for the capital stock in public sector and bias the results. Another possibility is that there is a segment of labor force in public sector, whose work is important for the society, but it is low paid, leading to lower output.[15] We assume however, that the problem with declining efficiency in public sector is most likely real. Further research will be needed to explore its true scope and causes. After identifying the source of the problem, corresponding remedial measures will be needed.

Conclusion

We present estimates of supply systems for Slovakia, each consisting of a production function and relative demand function and construct annual efficiency gains for private and public sector based on estimated systems. Main findings are threefold:

1. We find that the differences of production process in public and private sector are so great that they need to be modelled separately.

2. The assumptions about technical progress impact the results greatly. Use of single exponential trend leads to elasticity of substitution equal to one in private sector (Cobb-Douglas function) and zero in public sector (Leontieff function). The world of producing output however seems to be much more colourful than that. If individual efficiency trends are introduced, the elasticity of substitution for both sectors is significantly greater than zero, but below 0.5. Therefore, the

[15] We assume here that the corresponding small part of output in the public sector is defined as a sum of costs, rather than by the pricing in the market.

use of a CES function is required. Models using Box-Cox transformations are superior to models with exponential trend, as the Schwartz criterion shows that the improvement in fit outweighs the loss of efficiency caused by additional parameters. Thus, application of single exponential trend is too restrictive.

3. The efficiency gains (annual absolute increases of labour and capital efficiency derived from the model with Box-Cox transformation) increase or converge to a positive constant in the private sector, whereas they are strictly decreasing in the public sector. The results for public sector indicate either a significant slowdown in efficiency that needs to be addressed or a problem with measurement.

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Appendix

Derivation of estimation specification for relative demand with trends in this appendix, we generalize Oaxaca for the case with separate trends using Box-Cox transformation. CES function has the form

$$Q=\bar{Q}A\left[a\check{L}^{\rho}+(1-a)\check{K}^{\rho}\right]^{\frac{1}{\rho}} \tag{1A}$$

$$\text{where } \rho=\frac{\sigma-1}{\sigma},\ \check{K}=\left(\frac{K}{\bar{K}}\right)exp\left\{\frac{\bar{t}\gamma_1}{\lambda_1}\left[\left(\frac{t}{\bar{t}}\right)^{\lambda_1}-1\right]\right\} \text{ and} \tag{2A}$$

$$\check{L}=\left(\frac{L}{\bar{L}}\right)exp\left\{\frac{\bar{t}\gamma_2}{\lambda_2}\left[\left(\frac{t}{\bar{t}}\right)^{\lambda_2}-1\right]\right\} \tag{3A}$$

Marginal products for transformed inputs are

$$\frac{\partial Q}{\partial \check{K}}=(1-a)\check{K}^{\rho-1}C \text{ and } \frac{\partial Q}{\partial \check{L}}=a\check{L}^{\rho-1}C\text{, where } C=(\bar{Q}A)^{\rho}Q^{1-\rho}.$$

Marginal products for actual inputs are according to chain rule

$$\frac{\partial Q}{\partial K}=\frac{\partial Q}{\partial \check{K}}\frac{\partial \check{K}}{\partial K}=(1-a)\check{K}^{\rho-1}C\,exp\left\{\frac{\bar{t}\gamma_1}{\lambda_1}\left[\left(\frac{t}{\bar{t}}\right)^{\lambda_1}-1\right]\right\}\frac{1}{\bar{K}} \text{ and}$$

$$\frac{\partial Q}{\partial L} = a\,\check{L}^{\rho-1}\,C\,exp\left\{\frac{\bar{t}\,\gamma_2}{\lambda_2}\left[\left(\frac{t}{\bar{t}}\right)^{\lambda_2}-1\right]\right\}\frac{1}{\bar{L}}\,.$$

The ratio of marginal products is equal to the ratio of respective prices (wages and user costs). *C* cancels out, ratio of transformed inputs rearranged

$$\frac{w}{q} = \frac{a}{1-a}\left(\frac{\check{K}}{\check{L}}\right)^{1-\rho}\frac{exp\left\{\frac{\bar{t}\,\gamma_2}{\lambda_2}\left[\left(\frac{t}{\bar{t}}\right)^{\lambda_2}-1\right]\right\}\frac{1}{\bar{L}}}{exp\left\{\frac{\bar{t}\,\gamma_1}{\lambda_1}\left[\left(\frac{t}{\bar{t}}\right)^{\lambda_1}-1\right]\right\}\frac{1}{\bar{K}}} \tag{4A}$$

$$\left(\frac{\check{K}}{\check{L}}\right) = \left(\frac{K}{L}\right)\left(\frac{\bar{K}}{\bar{L}}\right)^{-1}\frac{exp\left\{\frac{\bar{t}\,\gamma_1}{\lambda_1}\left[\left(\frac{t}{\bar{t}}\right)^{\lambda_1}-1\right]\right\}}{exp\left\{\frac{\bar{t}\,\gamma_2}{\lambda_2}\left[\left(\frac{t}{\bar{t}}\right)^{\lambda_2}-1\right]\right\}} =$$

$$\left(\frac{1-a}{a}\right)^{\frac{1}{1-\rho}}\left(\frac{w}{q}\right)^{\frac{1}{1-\rho}}\left\{\frac{exp\left\{\frac{\bar{t}\,\gamma_1}{\lambda_1}\left[\left(\frac{t}{\bar{t}}\right)^{\lambda_1}-1\right]\right\}}{exp\left\{\frac{\bar{t}\,\gamma_2}{\lambda_2}\left[\left(\frac{t}{\bar{t}}\right)^{\lambda_2}-1\right]\right\}}\right\}^{\frac{1}{1-\rho}}\left(\frac{\bar{K}}{\bar{L}}\right)^{-\frac{1}{1-\rho}}$$

$$\left(\frac{K}{L}\right) = \left(\frac{1-a}{a}\right)^{\frac{1}{1-\rho}}\left(\frac{w}{q}\right)^{\frac{1}{1-\rho}}\left\{\frac{exp\left\{\frac{\bar{t}\,\gamma_1}{\lambda_1}\left[\left(\frac{t}{\bar{t}}\right)^{\lambda_1}-1\right]\right\}}{exp\left\{\frac{\bar{t}\,\gamma_2}{\lambda_2}\left[\left(\frac{t}{\bar{t}}\right)^{\lambda_2}-1\right]\right\}}\right\}^{\frac{1}{1-\rho}-1}\left(\frac{\bar{K}}{\bar{L}}\right)^{1-\frac{1}{1-\rho}}$$

Because $1/(1-\rho) = \sigma$, last equation can be reparametrized as

$$\left(\frac{K}{L}\right) = \left(\frac{1-a}{a}\right)^{\sigma}\left(\frac{\bar{K}}{\bar{L}}\right)^{1-\sigma}\left(\frac{w}{q}\right)^{\sigma}\left\{\frac{exp\left\{\frac{\bar{t}\,\gamma_1}{\lambda_1}\left[\left(\frac{t}{\bar{t}}\right)^{\lambda_1}-1\right]\right\}}{exp\left\{\frac{\bar{t}\,\gamma_2}{\lambda_2}\left[\left(\frac{t}{\bar{t}}\right)^{\lambda_2}-1\right]\right\}}\right\}^{\sigma-1} \tag{5A}$$

After merging constant terms (and labeling them C_2) and taking logarithms we estimate

$$log\left(\frac{K}{L}\right)=\log C_2+\sigma log\left(\frac{w}{q}\right)+(\sigma-1)\left\{\frac{\bar{t}\gamma_1}{\lambda_1}\left[\left(\frac{t}{\bar{t}}\right)^{\lambda_1}-1\right]-\frac{\bar{t}\gamma_2}{\lambda_2}\left[\left(\frac{t}{\bar{t}}\right)^{\lambda_2}-1\right]\right\} \quad (6A)$$

The aforementioned equation is static, but its estimates indicate that the volumes of production factors do not adjust to their prices in the same period, a dynamic specification is needed. In an ideal case, we would apply the error correcting methodology, but it has two major drawbacks: the resulting relation would be highly non-linear, and the emphasis would be shifted to short-term dynamics instead of long-term relationships. The reason is that the left-hand side variable would enter the specification in differences and differencing works as a filter suppressing long run information. We thus chose the simple Koyck transformation instead. The transformation consists of introducing the lagged value of left-hand variable into the right-hand side of the equation. Assume a dynamic relationship $x_t=c_0+c_1x_{t-1}+c_2z_t$, then the long run solution is $x^*=\frac{c_0}{1-c_1}+\frac{c_2}{1-c_1}z^*$, where $\frac{c_2}{1-c_1}$ is the long term elasticity of x to z. If this long run elasticity shall be equal to some value, say μ, the parameter c_2 would have to be restricted and the original specification shall have the form $x_t=c_0+c_1x_{t-1}+\mu(1-c_1)z_t$. Similarly, taking into account that the long-term elasticity of ratio of volumes of production factors to ratio of their price shall be equal to σ, we can dynamize the specification (5A) as follows:

$$\begin{aligned}log\left(\frac{K}{L}\right)&=\log C_3+\nu log\left(\frac{K_{t-1}}{L_{t-1}}\right)+(1-\nu)\sigma log\left(\frac{w}{q}\right)+\\&+(1-\nu)(\sigma-1)\left\{\frac{\bar{t}\gamma_1}{\lambda_1}\left[\left(\frac{t}{\bar{t}}\right)^{\lambda_1}-1\right]-\frac{\bar{t}\gamma_2}{\lambda_2}\left[\left(\frac{t}{\bar{t}}\right)^{\lambda_2}-1\right]\right\}\end{aligned} \quad (7A)$$

In this final specification, we have introduced a different constant C_3 and omitted the subscript t at variables to make the equation less cluttered.