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Article
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Standard-Nutzungsbedingungen:
An Unified Consumption and Production Model for a Closed Economy

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Abstract: The article presents an unified consumption and production model for a closed economy.

Keywords: consumption, production, utility

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1. Introduction

Let consider n goods: G₁, ..., Gₙ whose elasticity of utility is constant, their prices being p₁, ..., pₙ. For a consumer whose available income is V, the utility function corresponding to the consumption of xₚ units of good Gₚ, p=1, n: \( U(x_1, ..., x_n) = Ax_1^{\alpha_1}x_2^{\alpha_2}...x_n^{\alpha_n} \) where \( \alpha_p \) is the elasticity of utility in relation to the good Gₚ, and A is a positive constant.

The issue of maximizing the utility relative to the restriction: \( \sum_{k=1}^{n} p_k x_k \leq V \) is:

\[
\begin{align*}
\max U(x_1, ..., x_n) \\
\sum_{k=1}^{n} p_k x_k \leq V \\
x_1, ..., x_n \geq 0
\end{align*}
\]

Considering the Lagrangeian:

\[
\Phi(x_1, ..., x_n, \lambda) = U(x_1, ..., x_n) + \lambda \left( \sum_{k=1}^{n} p_k x_k - V \right)
\]

the maximum condition with restrictions must satisfy:

\[
\begin{align*}
\frac{\partial \Phi}{\partial x_j} &= \frac{\partial U}{\partial x_j} + \lambda p_j = 0, j = 1, n \\
\frac{\partial \Phi}{\partial \lambda} &= \sum_{j=1}^{n} p_j x_j - V = 0 \\
-\sum_{j=1}^{n} \frac{\alpha_j Ax_1^{\alpha_1}x_2^{\alpha_2}...x_n^{\alpha_n}}{\lambda} x_j - V &= 0 \iff \frac{Ax_1^{\alpha_1}x_2^{\alpha_2}...x_n^{\alpha_n}}{\lambda} \sum_{j=1}^{n} \alpha_j + V = 0
\end{align*}
\]

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\[
A x_1^{\alpha_1} \cdots x_n^{\alpha_n} \sum_{j=1}^{n} \alpha_j = \lambda V.
\]

By re-replacing, we get the optimal solution to the problem:

\[
x_j^* = \frac{\alpha_j V}{p_j \sum_{j=1}^{n} \alpha_j}, \quad j = 1, n
\]

Let us also consider a producer with a number of K capital units, having the price of \(p_K\) and L workers whose hourly wage is \(w\) for a working time \(t\). If the elasticity of production in relation to capital and labor are constant, the function of production is: \(Q(t,K,L) = CK^\beta L^\gamma\) where \(\beta\) and \(\gamma\) are the elasticities of production in relation to the capital, respectively the labor, \(C\) being a constant.

Total cost of production: \(CT = p_K K + twL\) leads to a gross profit corresponding to a sales price \(p\):

\[
\pi(t,K,L) = pQ(t,K,L) - CT = CpK^\beta L^\gamma - p_K K - twL.
\]

For a given production \(Q_0\), the profit maximization condition returns to minimizing the total cost, so to the problem:

\[
\begin{align*}
&\min (p_K K + twL) \\
&\text{subject to: } CK^\beta L^\gamma = Q_0
\end{align*}
\]

Considering the Lagrangeian:

\[
\Phi(x_1,\ldots, x_n, \lambda) = p_K K + twL + \lambda (CK^\beta L^\gamma - Q_0)
\]

the minimum condition with restrictions must satisfy:

\[
\begin{align*}
\frac{\partial \Phi}{\partial K} &= p_k + \lambda \beta CK^\beta - L^\gamma = 0 \\
\frac{\partial \Phi}{\partial L} &= tw + \lambda \gamma CK^\beta - L^\gamma = 0 \\
\frac{\partial \Phi}{\partial \lambda} &= CK^\beta L^\gamma - Q_0 = 0
\end{align*}
\]

\[
\begin{align*}
L^* &= Q_0 \left( \frac{p_K}{C} \right)^{\beta \gamma} \\
K^* &= Q_0 \left( \frac{p_K}{C} \right)^{-\gamma \beta}
\end{align*}
\]

and the minimum total cost:

\[
CT^* = Q_0 \left[ p_k \left( \frac{p_K \gamma}{C \beta tw} \right)^{\beta \gamma} + tw \left( \frac{p_K \gamma}{C \beta tw} \right)^{-\gamma \beta} \right].
\]

The maximum profit is:
\[ \pi(t,K^*,L^*) = pQ_0 - \frac{Q_0}{C} \left( p_K \left( \frac{p_K \gamma}{\beta tw} \right)^\beta + tw \left( \frac{p_K \gamma}{\beta tw} \right)^\beta \right) \]

2. The Model

Suppose there are a number of \( n \) firms: \( F_1, ..., F_n \) each having a number of \( L_i \) employees, \( i=1, n \) where we will include, for simplification, the entrepreneur of the firm. Let \( w_i \) - the average hourly wage for \( F_i \), \( t_i \) - the working time during the analysis period in \( F_i \). We will also assume that \( F_i \) produces a single good (of constant elasticity): \( G_i \) whose sales price is \( p_i \).

From the total revenue received, each employee pays a tax quota to the state budget \( \chi \).

For health insurance, pensions and other services that will later be paid back to employees, they pay a share \( \xi \) of the salary received. Let us consider the providers of these services (a single service \( G_{n+j} \) for each firm) as being the firms \( F_{n+1}, ..., F_{n+m} \) each having \( L_{n+j} \), \( j=1, m \) employees (including the entrepreneur), with \( w_{n+j} \) - the average hourly wage corresponding to the company \( F_{n+j} \) and \( t_{n+j} \) - the working time worked during the analysis period in \( F_{n+j} \). The service price offered by \( F_{n+j} \) will be \( p_{n+j} \).

Therefore, the tax paid by each employee will be: \( T_i = \chi w_i t_i \), \( i=1, n+m \) and for public services: \( T_i = \xi w_i t_i \), \( i=1, n+m \).

The revenues available to \( F_i \) staff are therefore (for each individual employee): \( V_i = (1-\chi-\xi)w_i t_i \), \( i=1, n+m \).

On the other hand, the amount of salaries received by service providers comes from the share \( \xi \) therefore: \( \xi \sum_{i=1}^{n+m} w_i t_i L_i = \sum_{j=1}^{m} w_{n+j} t_{n+j} L_{n+j} \) or: \( \sum_{j=1}^{m} w_{n+j} t_{n+j} L_{n+j} = \frac{\xi}{1-\xi} \sum_{i=1}^{n} w_i t_i L_i \).

The entrepreneur of \( F_i \), \( i=1, n+m \) will allocate the profits made for investments that will be considered as goods produced by firms. Let us consider the output of \( F_i \) as: \( Q_i(t,K_i,L_i) = C_i K_i^\beta L_i^\gamma \) where \( \beta \) and \( \gamma \) represent the constantly assumed elasticities of production relative to \( K_i \), respectively \( L_i \), \( C_i \) - positive constant. At a price of capital \( p_{K_i} \), the total cost of production in \( F_i \) becomes: \( CT_i = p_{K_i} K_i + t_i w_i L_i \).

Therefore, at a sale price \( p_i \) of the \( G_i \) asset, \( F_i \)'s profit is:

\[ \pi_i = p_i Q_i(t,K_i,L_i) - p_{K_i} K_i - t_i w_i L_i = p_i C_i K_i^\beta L_i^\gamma - p_{K_i} K_i - t_i w_i L_i \]

The \( F_i \)'s firm's entrepreneur income will therefore be just that \( \pi_i, i=1, n+m \).

Let considering the set \( S \) of social assistants (pensioners, people without income etc.) with a number of \( M \) people whose incomes represents a share \( \rho \) of the taxes paid by employees (the remainder being allocated to government consumption, public works etc.). Their income will therefore be:

\[ \rho(\text{TP}_b + \text{TS}_b) = \rho \gamma \sum_{i=1}^{n+m} w_i t_i L_i \] and the average income per person:

\[ \frac{\rho \gamma \sum_{i=1}^{n+m} w_i t_i L_i}{M} \]
In the following, we will consider that the utility function of any employee of a production, service or social assistance company will be the same for all consumers within the category (it may be different from company to company – as an example, the utility of books is different for employees of an educational establishment and another for meat producers). In addition, we will assume that all the production of a company will be sold.

Consider the utility functions for an employee of $F_i$: $\bar{U}_i(x_{i1}, \ldots, x_{i,n+m}) = A_i x_{i1}^{a_1} \ldots x_{i,n+m}^{a_{n+m}}, \ i=1,n+m$ where $x_{ij}$ represents the quantity of good $G_j$ consumed by an employee of $F_i$ and for social assistants: $\bar{U}_s(x_1, \ldots, x_{n+m}) = A x_1^{a_1} \ldots x_{n+m}^{a_{n+m}}$ where $x_j$ represents the amount of good $G_j$ consumed by a social assistant.

The utility functions of the entrepreneurs in the investment activity will be:

$$\tilde{U}_j(y_{i1}, \ldots, y_{in}) = B_j y_{i1}^{\delta_1} \ldots y_{in}^{\delta_n}, \ i=1,n+m$$

where $y_{ij}$ represents the amount of good $G_j$ consumed by the $F_i$'s entrepreneur.

Every employee and entrepreneurs want to maximize their utilities in the context of disposable income, so problems arise:

\begin{equation}
\begin{align*}
\max_{n+m} \bar{U}_i(x_{i1}, \ldots, x_{i,n+m}) &= A_i x_{i1}^{a_1} \ldots x_{i,n+m}^{a_{n+m}} \\
\sum_{k=1}^{n+m} p_k x_{ik} \leq (1 - \chi - \xi) w_i t_i & \quad \text{for company employees } F_i, \ i=1,n+m; \\
x_{i1}, \ldots, x_{i,n+m} &\geq 0
\end{align*}
\end{equation}

\begin{equation}
\begin{align*}
\max_{n+m} \bar{U}_s(x_1, \ldots, x_{n+m}) &= A x_1^{a_1} \ldots x_{n+m}^{a_{n+m}} \\
\sum_{k=1}^{n+m} p_k x_k \leq \frac{\rho \chi}{M} \sum_{i=1}^{n+m} w_i t_i & \quad \text{for social assistants;} \\
x_{1}, \ldots, x_{n+m} &\geq 0
\end{align*}
\end{equation}

\begin{equation}
\begin{align*}
\max \tilde{U}_j(y_{i1}, \ldots, y_{in}) &= B_j y_{i1}^{\delta_1} \ldots y_{in}^{\delta_n} \\
\sum_{k=1}^{n} p_k y_{ik} \leq p_j C_i K_i^{\gamma_i} L_i^{\gamma_i} - p_j K_i - t_i w_i L_i & \quad \text{for entrepreneurs.} \\
y_{i1}, \ldots, y_{in} &\geq 0
\end{align*}
\end{equation}

It follows from the above that the optimum quantities of products are:

- $x_{ij}^* = \frac{\alpha_j (1 - \chi - \xi) w_i t_i}{p_j \sum_{k=1}^{n+m} \alpha_k}, j=1,n+m$ - for company employees $F_i, \ i=1,n+m$;
- $x_j^* = \frac{\alpha_j \frac{\rho \chi}{M} \sum_{i=1}^{n+m} w_i t_i}{p_j \sum_{k=1}^{n+m} \alpha_k}, j=1,n+m$ - for social assistants;
- $y_{ij}^* = \frac{\delta_j p_j C_i K_i^{\gamma_i} L_i^{\gamma_i} - p_j K_i - t_i w_i L_i}{p_j \sum_{k=1}^{n+m} \delta_{ik}}, j=1,n+m$ - for entrepreneurs.
Therefore, the amount of required $G_j$ needed is:

$$Q_j = \sum_{i=1}^{m+n} L_i x_{ij}^* + M \sum_{i=1}^{m+n} y_{ij}^* =$$

$$\frac{1 - \chi - \xi}{p_j} \sum_{i=1}^{m+n} L_i \alpha_j w_{ij} t_i + \frac{\rho \chi \alpha_j (m + n) \sum_{i=1}^{m+n} w_i t_i L_i}{p_j} + \frac{1}{p_j} \sum_{i=1}^{m+n} \delta_j p_i C_i K_i^{\gamma_j} L_i^{\gamma_j} - p_k K_i - t_i w_i L_i, \quad j = l, n + m .$$

Returning to the problem of maximizing $F_j$'s profit for the quantity $Q_j$ we have:

$$\begin{cases}
\text{min} \left\{ p_k K_i + t_i w_i L_i \right\} \\
C_j K_i^{\gamma_j} L_i^{\gamma_j} = Q_j , \quad j = l, n + m \\
K_j L_j \geq 0
\end{cases}$$

from where:

$$\begin{cases}
L_j^* = \frac{Q_j}{C_j \left( \frac{p_k}{\beta_j t_j w_j} \right)^{\gamma_j}} \\
K_j^* = \frac{Q_j}{C_j \left( \frac{p_k}{\beta_j t_j w_j} \right)^{\gamma_j}}
\end{cases}$$

Noting for simplicity:

$$s_i = w_i t_i, \quad r_i = \frac{\alpha_j s_i}{n + m}, \quad u_j = \frac{\rho \chi (m + n) \alpha_j}{n + m}, \quad v_i = \frac{\delta_j p_i C_i}{n + m}, \quad z_i = \frac{p_k}{n + m}, \quad f_i = \frac{s_j}{n + m},$$

$$g_j = \left( \frac{p_k}{\beta_j t_j w_j} \right)^{\gamma_j} \Rightarrow p_k \frac{\gamma_j}{\beta_j t_j w_j} = g_j \left( C_j p_j \right)^{\gamma_j}$$

from where:

$$\begin{cases}
L_j^* = \frac{g_j}{C_j p_j} \left( 1 - \chi - \xi \right) \sum_{i=1}^{m+n} r_i L_i^* + u_j \sum_{i=1}^{m+n} s_i L_i^* + \sum_{i=1}^{m+n} \left( v_i K_i^{\gamma_j} L_i^{\gamma_j} - z_i K_i^{\gamma_j} - f_i L_i^{\gamma_j} \right) \\
K_j^* = \frac{g_j}{C_j p_j} \left( 1 - \chi - \xi \right) \sum_{i=1}^{m+n} r_i L_i^* + u_j \sum_{i=1}^{m+n} s_i L_i^* + \sum_{i=1}^{m+n} \left( v_i K_i^{\gamma_j} L_i^{\gamma_j} - z_i K_i^{\gamma_j} - f_i L_i^{\gamma_j} \right)
\end{cases}$$

we find that:

$$\begin{cases}
L_j^* = \sum_{i=1}^{m+n} g_j \left( 1 - \chi - \xi \right) r_i L_i^* + u_j \sum_{i=1}^{m+n} s_i L_i^* + \sum_{i=1}^{m+n} \left( v_i K_i^{\gamma_j} L_i^{\gamma_j} - z_i K_i^{\gamma_j} - f_i L_i^{\gamma_j} \right) \\
K_j^* = \sum_{i=1}^{m+n} g_j \left( 1 - \chi - \xi \right) r_i L_i^* + u_j \sum_{i=1}^{m+n} s_i L_i^* + \sum_{i=1}^{m+n} \left( v_i K_i^{\gamma_j} L_i^{\gamma_j} - z_i K_i^{\gamma_j} - f_i L_i^{\gamma_j} \right)
\end{cases}$$

or, in other words:

$$\begin{cases}
L_j^* = \sum_{i=1}^{m+n} g_j \left( 1 - \chi - \xi \right) r_i s_i + u_j \sum_{i=1}^{m+n} g_j v_i K_i^{\gamma_j} L_i^{\gamma_j} - \sum_{i=1}^{m+n} g_j z_i K_i^{\gamma_j} \\
K_j^* = \sum_{i=1}^{m+n} g_j \left( 1 - \chi - \xi \right) r_i s_i + u_j \sum_{i=1}^{m+n} g_j v_i K_i^{\gamma_j} L_i^{\gamma_j} - \sum_{i=1}^{m+n} g_j z_i K_i^{\gamma_j}
\end{cases}$$
Noting again:

- \( Y_{1,ij} = g_j \left( t_j (1 - \chi - \xi) + u_j s_i - f_i \right) \)
- \( Y_{2,ij} = g_j v_{ij} \)
- \( Y_{3,ij} = g_j z_i \)
- \( Y_{4,ij} = \gamma_j \left( C_j p_j \beta_j \right)^{\gamma_j - 1} \left( t_j (1 - \chi - \xi) + u_j s_i - f_i \right) \)
- \( Y_{5,ij} = \gamma_j \left( C_j p_j \beta_j \right)^{\gamma_j - 1} v_{ij} \)
- \( Y_{6,ij} = \gamma_j \left( C_j p_j \beta_j \right)^{\gamma_j - 1} z_i \)

we obtain:

\[
\begin{align*}
L_j &= \sum_{i=1}^{m+n} Y_{1,ij} L^*_i + \sum_{i=1}^{m+n} Y_{2,ij} K^*_i L^*_i - \sum_{i=1}^{m+n} Y_{3,ij} K^*_i, \\
K^*_i &= \sum_{j=1}^{m+n} Y_{4,ij} L^*_i + \sum_{j=1}^{m+n} Y_{5,ij} K^*_i L^*_i - \sum_{j=1}^{m+n} Y_{6,ij} K^*_i
\end{align*}
\]

The system solution will provide the optimal number of employees of each firm as well as the required capital.

On the other hand, provided that: \( \sum_{j=1}^{m+n} w_{n+j} l_{n+j} L_{n+j} \) follows:

\[
\sum_{j=1}^{m+n} w_{n+j} l_{n+j} L_{n+j}^* = \frac{\xi}{1 - \xi} \sum_{i=1}^{n} w_i t_i L^*_i.
\]

By replacing the above optimal solutions, we obtain the link between the two quotas (the only ones that are imposed at government level): \( \chi \) (tax) and \( \xi \) - for health insurance, pensions and other services.

3. References
